Feb. 26, 2014

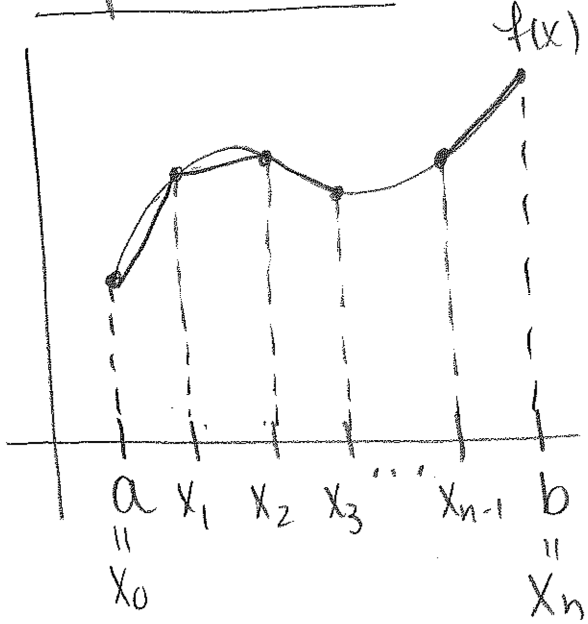
Approximating Integrals

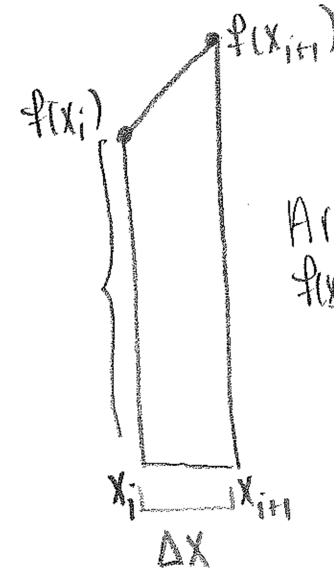
$$\int_0^1 e^{x^2}\,dx \qquad \int_{-1}^1 \sqrt{1+x^3}\,dx$$

} can't find an exact antiderivative

- When you are collecting data, you will have to numerically approx area under curve.

Trapezoid Rule

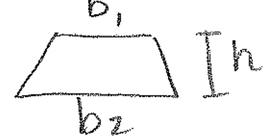




Area is $\frac{f(x_i)+f(x_{i+1})}{2}\cdot \Delta x$

$\Delta x = \frac{b-a}{n}$

area of trapezoid is



$\frac{b_1+b_2}{2}\cdot h$

$$\boxed{\underbrace{T_n}_{\text{estimation using } n \text{ trapezoids}} = \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right)}$$

$$= \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \ldots + \frac{f(x_{n-1})+f(x_n)}{2}\Delta x$$

$$T_n = \frac{h}{2}\left(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n\right)$$

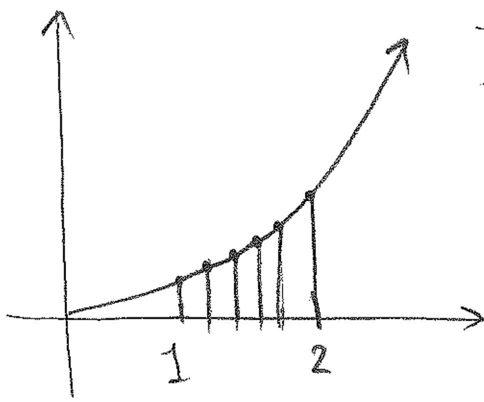
Webwork Notation:

Let $y_i = f(x_i)$

$$T_n = \frac{\overset{\Delta x}{h}}{2}\left(\underbrace{y_0}_{f(x_0)} + \underbrace{2\sum_{j=1}^{n-1} y_j}_{2\sum_{j=1}^{n-1} f(x_j)} + \underbrace{y_n}_{f(x_n)}\right)$$

Ex: Use trapezoid rule w/ $n=5$ to estimate

$\int_1^2 1/x \, dx$ so $f(x) = 1/x$ $\qquad \Delta x = \frac{2-1}{5} = \frac{1}{5}$



i	x_i	$f(x_i) = y_i$
0	1	1
1	6/5	5/6
2	7/5	5/7
3	8/5	5/8
4	9/5	5/9
5	2	1/2

$$T_5 = \frac{1/5}{2}\left(1 + 2\left(\frac{5}{6}\right) + 2\left(\frac{5}{7}\right) + 2\left(\frac{5}{8}\right) + 2\left(\frac{5}{9}\right) + \frac{1}{2}\right) \approx 0.695635$$

$$\frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5\right)$$

Error - How close is our estimate?

$$\int_1^2 \frac{1}{x} dx = \ln x \Big|_1^2 = \ln 2 - \ln 1 = 0.693147$$

$|\text{error}_{T_n}| =$ actual - approximated

$= .693147 - .695635 \approx -.002488$

Error Bounds for Trapezoid Rule

A rule that tells us the largest the area could be.

terminology

M_2 = absolute max of $|f''(x)|$ on $[a, b]$

$$|\text{error}_{T_n}| \le \frac{M_2(b-a)^3}{12n^2}$$

Example: $\int_1^2 \frac{1}{x}\,dx$

$$|\text{error}_{T_5}| \le \frac{M_2(2-1)^3}{12(5)^2} = \frac{M_2}{12 \cdot 5^2}$$

$$\le \frac{2}{12 \cdot 5^2} \approx .00667$$

What is M_2?

$f(x) = \frac{1}{x}$

$f'(x) = -1/x^2$

$f''(x) = \underbrace{\frac{2}{x^3}}_{\text{always pos on } [1,2],}\Big\}$ max

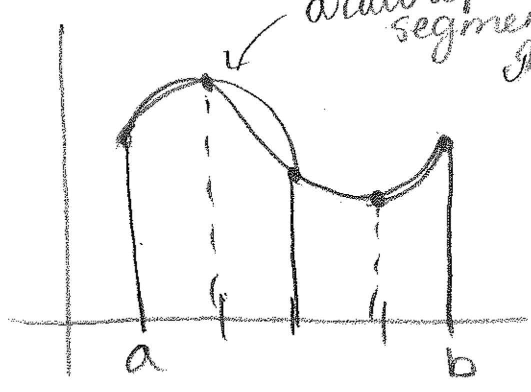
$f'''(x) = -\frac{6}{x^4}$

never zero, always neg

So $f''(1) = 2$ is abs max on $[1, 2]$

Simpson's Rule - estimate w/ parabolas

n is even



draw a parabolic segment that goes thru

$(x_{i-1}, f(x_{i-1}))$

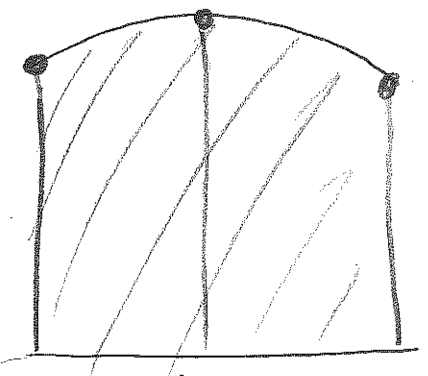
$(x_i, f(x_i))$

$(x_{i+1}, f(x_{i+1}))$

one segment

find equation of parabola segment

$y = Ax^2 + Bx + C$



Solve for A, B, C using info about the three coordinates we know it goes thru.

Formula for calculating area w/ Simpson's Rule:

n is even, $n = \#$ of strips

$$\boxed{S_n = \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)}$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

$$= \frac{h}{3}\left(y_0 + 4\sum y_{odd} + 2\sum y_{even} + y_n\right)$$

Example: Estimate $\int_0^{\pi/2} \sin x\,dx$ using Simpson's method w/ $n=6$.

$$h = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$$

i	x_i	$y_i = f(x_i)$
0	0	0
1	$\pi/12$	.2588
2	$2\pi/12$	$1/2$
3	$3\pi/12$	$\sqrt{2}/2$
4	$4\pi/12$	$\sqrt{3}/2$
5	$5\pi/12$	.9659
6	$\pi/2$	1

$$S_6 = \frac{\left(\frac{\pi}{12}\right)}{3}\left(0 + 4(.2588) + 2\left(\tfrac{1}{2}\right) + 4\left(\tfrac{\sqrt{2}}{2}\right) + 2\left(\tfrac{\sqrt{3}}{2}\right) + 4(.9659) + 1\right)$$

$$= 1.0000263\ldots$$

Notice $\int_0^{\pi/2} \sin x\,dx = -\cos x \Big|_0^{\pi/2} = 1$

$|error_{S_6}| = |1 - 1.0000263...| \approx .0000263$

Error Bound For S_n:

M_4 = absolute max of $|f^{(4)}(x)|$ on $[a,b]$

$$|error_{S_n}| \leq \frac{M_4(b-a)^5}{180n^4}$$

Example: Find max error for $\int_0^{\pi/2} \sin x\,dx$ where $n=6$.

First find M_4: $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \underline{\underline{\sin x}}$

$\sin x$ never goes negative in $[0, \pi/2]$.

We know max is 1 (at $\pi/2$)

So $M_4 = 1$

$$|error_{S_6}| \leq \frac{1(\pi/2-0)^5}{180(6)^4} \approx \underbrace{0.00004099...}_{\text{error will not be greater than this.}}$$